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Linear vibrations of a rotating elastic beam with an attached point mass

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Abstract. The paper deals with linear vibrations of a thin elastic beam with an attached point mass which is clamped radially to the inside of a rigid wheel rotating at a constant angular velocity. Material damping is also taken into consideration. A partial differential equation of motion with dynamical boundary conditions is derived. It is proved that in a sense the attached point mass plays a destabilizing role for any values of the problem parameters. The problem of the determination of critical parameter values is reduced to a second-order Sturm-Liouville problem with a singularity. Asymptotic representations of the stability threshold, as the ratio between the radius of the wheel and the length of the beam goes either to 0 or to $+\infty$, are obtained. The dynamical stability of the rectilinear shape of the beam is studied by means of the direct Lyapunov method. A non-standard method of construction of Chetayev's function is proposed.

Key words: asymptotic expansion, Chetayev functions, rotating beam, stability

1. Introduction

We will consider a rather simple mechanical system which nevertheless attracts the attention of many scientists in the field of mechanical engineering and also in the field of applied mathematics. There are several reasons for that. First, this system exhibits many common features that an engineer faces when modelling a real system containing deformable rotating elements. While a famous Lagrange top was the main simplified model for the gyroscope theory, the system to be considered can be treated as a certain approximation to the theory of rotating shafts. Second, the above problem provides us with a lot of interesting physical phenomena both of a linear and nonlinear nature. The mathematical description of some of these phenomena can be so complicated that the solution of the corresponding problems may require quite sofisticated mathematical methods. We consider the motion of an elastic thin homogeneous beam of a linear mass density ρ clamped radially to the inside of a rigid wheel (See Figure 1). Let EI be the bending stiffness of the beam. Also, let a point mass m be attached at the other end of the beam. Let us suppose that the wheel rotates uniformly about its center in a horizontal plane at a constant angular velocity ω. We can thus neglect the action of gravity forces. The system has an obvious relative equilibrium position when the beam preserves its rectilinear shape. The present work is aimed at deriving linear equations of motion of the above system and finding appropriate stability conditions of the trivial shape. The problem like that for the beam without a point mass was studied in many papers. See, for instance [1–12]. But the problem involving a point mass should be treated more carefully. Publications concerning the above subject are much smaller in number. Here it is worth citing the following papers [13–15]. The authors of [12] proved that a rotating beam without a point mass looses stability

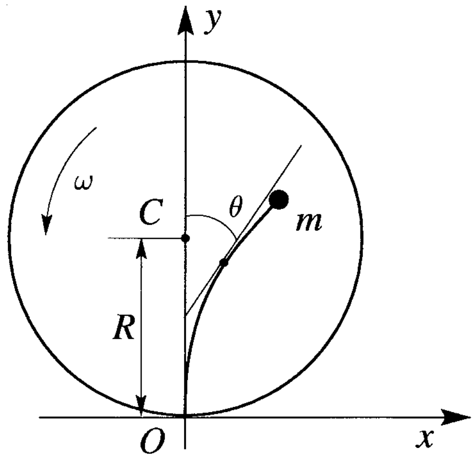

Figure 1. Rotating beam with a point mass

if the magnitude of the angular velocity is large enough for any values of the ratio between the length of the beam and the radius of the wheel. From a physical point of view, the point mass seems to play a stabilizing role if only the length of the beam is greater than the radius of the wheel. This is because the centrifugal forces acting at the point mass try to rectify the beam. We will show, however, that also in this case the beam looses stability when the angular velocity is high enough. Another feature that makes the system under consideration different from those studied in [1–12] is material damping. We can symbolically designate the number of degrees of freedom of the system as $\infty + 1$. Formally, the point mass provides us with an additional degree of freedom on which the energy does not dissipate. That is why it is difficult to prove asymptotic stability of the trivial shape using the standard approach and the total energy as a Lyapunov function. Nevertheless, the problem with the point mass and the one without it have very much in common. We will obtain some asymptotic expressions for the so-called buckling equation (stability threshold), which are quite similar for both cases.

2. Equations of motion

Let the length of the beam be equal to l and the radius of the wheel be equal to R. The motion of the whole system can be described by two functions $x(s,t)$ and $y(s,t)$ which are the coordinates of a current point of the beam in the rotating frame *Oxy* (See Figure 1), where $0 \le s \le l$ is the current length and $0 \le t < +\infty$ is the current time. Under the assumption that the beam is non-stretchable, those two functions cannot be treated as independent, since the following relation holds between them

$$(x'(s,t))^2 + (y'(s,t))^2 = 1$$

for any $0 \le s \le l$ and $t \ge 0$.

Consequently, we have to introduce a new independent distributed Lagrangian coordinate. The angle $\theta(s,t)$ between the tangent to a current point of the beam and the axis Oy can play the role of such a coordinate. Then the magnitudes x and y can be explicitly expressed through θ as follows

$$x(s,t) = \int_0^s \sin\theta(\xi,t)\,\mathrm{d}\xi, \quad y(s,t) = \int_0^s \cos\theta(\xi,t)\mathrm{d}\xi.$$

Since one end of the beam is clamped, the function $\theta(s,t)$ has to obey the boundary condition $\theta(0,t) = 0$. We can write the following expression for the kinetic energy of the beam

$$
\begin{aligned}
T &= \frac{\rho}{2}\int_0^l [(\dot{x} + \omega(R-y))^2 + (\dot{y} + \omega x)^2]\mathrm{d}s \\
&+ \frac{m}{2}[(\dot{x}(l,t) + \omega(R - y(l,t)))^2 + (\dot{y}(l,t) + \omega x(l,t))^2].
\end{aligned}
$$

Here and below an overdot means a partial derivative with respect to a temporal variable while a prime denotes a partial derivative with respect to a spatial variable.

Obviously, the trivial rectilinear shape of the beam $\theta(s,t) = 0$ is a relative equilibrium position. To derive linear equations of motion, we should rewrite the last expression in terms of θ and keep only quadratic terms with respect to θ and $\dot{\theta}$. Further, taking into account that in the linear approximation

$$
x(s,t) \sim \int_0^s \theta(\xi,t)\mathrm{d}\xi, \quad \text{and} \quad \theta(s,t) \sim x'(s,t)
$$

and, carrying out the following integration by parts,

$$
\int_0^l (R-s)\int_0^s \theta^2 \mathrm{d}\xi \ \mathrm{d}s = \int_0^l (R-s)\int_0^s x'^2 \mathrm{d}\xi \ \mathrm{d}s = -\frac{1}{2}\int_0^l (s-l)(2R-(s+l))x'^2\mathrm{d}s,
$$

we obtain the following quadratic expression for the kinetic energy of the beam

$$
\begin{aligned}
T &\sim \frac{\rho}{s}\int_0^l [\dot{x}^2 - \frac{\omega^2}{2}(s-l)(2R-(s+l)x'^2 + \omega^2 x^2]\ \mathrm{d}s \\
&+ \frac{m}{2}[\dot{x}^2(,t) + \omega^2(R-l)\int_0^l x'^2\mathrm{d}s + \omega^2 x^2(l,t)].
\end{aligned}
\tag{2.1}
$$

It is worth noting that we may use now $x(s,t)$ as a generalised coordinate and $\dot{x}(x,t)$ as a generalised velocity in the linear approximation.

Let us assume that the elastic beam under consideration obeys Hooke's law. The potential energy of elastic forces can be written as follows

$$
U = \frac{EI}{2}\int_0^l \theta'^2 \ \mathrm{d}s \sim \frac{EI}{2}\int_0^l x''^2\mathrm{d}s. \tag{2.2}
$$

If we deal with material damping in accordance with the Kelvin-Voigt model, the corresponding dissipative function reads

$$
\Phi = \Phi[\dot{\theta}] = kU[\dot{\theta}] = k\frac{EI}{2}\int_0^l \dot{\theta}'^2\mathrm{d}s \sim k\frac{EI}{2}\int_0^l \dot{x}''^2\mathrm{d}s, \tag{2.3}
$$

where k is the viscosity coefficient.

By neglecting the material damping, we can derive the desired equations of motion by means of Hamilton's variational principle

$$
\delta\int_{t_1}^{t_2}(T-U)\mathrm{d}t = 0.
$$

Consequently,

$$0 = \int_{t_1}^{t_2} \left\{ \rho \int_0^l \left[\dot{x}\delta\dot{x} - \frac{\omega^2}{2}(s-l)(2R-(s+l))x'\delta x' + \omega^2 x\delta x - \frac{EI}{\rho}x''\delta x'' \right] \mathrm{d}s \right. \\ \left. + m[\dot{x}(l,t)\delta\dot{x}(l,t) + \omega^2 \int_0^l x'\delta x' \mathrm{d}s + \omega^2 x(l,t)\delta x(l,t)] \right\} \mathrm{d}t \tag{2.4}$$

for any virtual displacements $\delta x(s,t)$ satisfying the following boundary conditions

$$\delta x(0,t) = \delta x'(0,t) = 0, \quad \delta x(s,t_1) = \delta x(s,t_2) = 0. \tag{2.5}$$

Besides boundary conditions (2.5), we should use also the geometric boundary conditions on the function $x(s,t)$ itself

$$x(0,t) = x'(0,t) = 0. \tag{2.6}$$

By integrating relation (2.4) and using boundary conditions (2.5) and (2.6), we obtain the following variational equation

$$0 = \int_{t_1}^{t_2} \left\{ \rho \int_0^l [-\ddot{x} + \frac{\omega^2}{2}((s-l)(2R-(s+l))x')' \right. \\ \left. + \omega^2 x - \frac{EI}{\rho}x^{\mathrm{IV}} - \frac{m\omega^2}{\rho}(R-l)x''] \right\} \delta x \mathrm{d}s\, \mathrm{d}t + \int_{t_1}^{t_2} \{[m(-\ddot{x}(l,t) \\ + \omega^2(x(l,t) + (R-l)x'(l,t))) + EIx'''(l,t)]\delta x(l,t) - EIx''(l,t)\delta x'(l,t)\}\mathrm{d}t. \tag{2.7}$$

Since the variations $\delta x(s,t), 0 < s < l$, $\delta x(l,t)$, $\delta x'(l,t)$, $t_1 < t < t_2$ can be chosen independently, using (2.7) we can derive a partial differential equation for the motion and boundary conditions at the free end. Thus, the equation of motion reads

$$\rho\left(\ddot{x} - \frac{\omega^2}{2}((s-l)(2R-(s+l))x')' - \omega^2 x\right) - EIx^{\mathrm{IV}} + m\omega^2(R-l)x'(l,t) = Q(s,t) \tag{2.8}$$

and the corresponding boundary conditions take the following form

$$x(0,t) = 0, \tag{2.9}$$

$$x'(0,t) = 0, \tag{2.10}$$

$$EIx''(l,t) = Q_1(l,t), \tag{2.11}$$

$$m(\ddot{x}(l,t) - \omega^2(x(l,t) + (R-l)x'(l,t))) - EIx'''(l,t) = Q_2(l,t), \tag{2.12}$$

where the influence of the material damping has been taking into account and the functions $Q(s,t)$, $Q_1(l,t)$, $Q_2(l,t)$ represent geneneralised forces caused by that material damping.

As follows from (2.3), in the linear approximation the dissipative function has the following form

$$\Phi = k\frac{EI}{2}\int_0^l \dot{x}''^2 \mathrm{d}s.$$

Therefore, the virtual work of the material-damping forces can be expressed as follows

$$\delta A(t) = -kEI\int_0^l \dot{x}''(s,t)\delta x''(s,t)\mathrm{d}s = \int_0^l Q(s,t)\delta x(s,t)\mathrm{d}s$$
$$+Q_1(l,t)\delta x'(l,t) + Q_2(l,t)\delta x(l,t).$$

Integrating the last relation by parts, we simply derive the necessary expressions for Q, Q_1, Q_2

$$Q(s,t) = -kEI\dot{x}^{\mathrm{IV}}, \quad Q_1(l,t) = -kEI\dot{x}''(l,t), \quad Q_2(l,t) = kEI\dot{x}'''(l,t),$$

which should be substituted in Equation (2.8) and the boundary conditions (2.11) and (2.12).

Now we can rewrite the obtained system of equations in dimensionless form by introducing dimensionless dependent and independent variables as follows

$$\sigma = s/l, \quad \tau = \omega t, \quad u = x/l.$$

Let us also introduce a set of independent dimensionless parameters

$$\alpha = \frac{R}{l}, \quad \epsilon = \frac{m}{\rho l}, \quad \lambda = \frac{\rho\omega^2 l^4}{EI}, \quad \beta = k\omega.$$

Then it is not very difficult to obtain the following system of equations

$$\ddot{u} + \frac{1}{2}(f(\alpha,\sigma)u')' - u + \lambda^{-1}(u^{\mathrm{IV}} + \beta\dot{u}^{\mathrm{IV}}) + \epsilon(\alpha-1)u'' = 0, \tag{2.13}$$

$$u(0,\tau) = 0, \tag{2.14}$$

$$u'(0,\tau) = 0, \tag{2.15}$$

$$\lambda^{-1}(u''(1,\tau) + \beta\dot{u}''(1,\tau)) = 0, \tag{2.16}$$

$$\epsilon(\ddot{u}(1,\tau) - u(1,\tau) - (\alpha-1)u'(1,\tau)) - \lambda^{-1}(u'''(1,\tau) + \beta\dot{u}'''(1,\tau)) = 0, \tag{2.17}$$

where $f(\alpha,\sigma) = (\sigma-\alpha)^2 - (1-\alpha)^2$

3. Lyapunov stability

First, it is worth noting that system (2.13)–(2.17) admits a Lagrangian setting. The Lagrangian reads as follows

$$\mathcal{L} = \mathcal{T} - \mathcal{V},$$

where $\mathcal{T}$ is an analogue of the kinetic energy

$$\mathcal{T} = \frac{1}{2}\left(\int_0^1 \dot{u}^2\mathrm{d}\sigma + \epsilon\dot{u}^2(1,\tau)\right) \tag{3.1}$$

and $\mathcal{V}$ is a dimensionless reduced potential energy (3.2)

$$\begin{aligned}\mathcal{V} &= \mathcal{V}_0 + \epsilon\mathcal{V}_1 = \frac{\lambda^{-1}}{2}\int_0^1 u''^2\mathrm{d}\sigma - \frac{1}{2}\int_0^1\left[\frac{1}{2}f(\alpha,\sigma)u'^2 + u^2\right]\mathrm{d}\sigma \\ &\quad -\frac{\epsilon}{2}\left[(\alpha-1)\int_0^1 u'^2\mathrm{d}\sigma + u^2(1,\tau)\right],\end{aligned} \tag{3.2}$$

where $\mathcal{V}_0$ means the expression for the reduced potential energy of the system without the point mass.

To investigate dynamical properties of the system under consideration, we need to define appropriate functional spaces which the corresponding variables belong to. Moreover, it is well known [16, 17] that for infinite-dimensional systems we can obtain different results on stability using different norms. In other words, we have to construct a suitable phase space. If no damping is present, we have an analogue of the total energy conservation law

$$\mathcal{H} = \mathcal{T} + \mathcal{V} \equiv h = \text{ const.} \tag{3.3}$$

In this case the situation is clear. The phase space should be organised in such a way that the functional (3.3) would be finite and continuous. If nevertheless material damping occurs, the requirements are more restrictive, since the dissipative function

$$\Phi = \frac{\lambda^{-1}\beta}{2}\int_0^1 \dot{u}''^2 \mathrm{d}\sigma \tag{3.4}$$

should be finite and continuous too.

The necessary functional spaces are described in the Appendix below.

Using lemmas formulated in the Appendix, we can easily prove boundedness and continuity of the potential and kinetic-energy functionals on the natural phase space $\mathbf{H}_0^2[0,1] \times \tilde{\mathbf{L}}_2[0,1]$.

Indeed,

$$\mathcal{T} = \frac{1}{2}\|\dot{u}\|_*^2.$$

If $0 < \alpha \le \frac{1}{2}$ (case I), the function $f(\alpha,\sigma)$ is negative and $\min_{\sigma\in[0,1]} f = -(1-\alpha)^2$, $\max_{\sigma\in[0,1]} f = 0$. If $\frac{1}{2} < \alpha \le 1$ (case II), the above function is alternating and $\min_{\sigma\in[0,1]} f = -(1-\alpha)^2$, $\max_{\sigma\in[0,1]} f = 2\alpha - 1$. Finally, if $\alpha > 1$ (case III), $f(\alpha,\sigma)$ is positive and $\min_{\sigma\in[0,1]} f = 0$, $\max_{\sigma\in[0,1]} f = 2\alpha - 1$.

Then for the functional of the potential energy (3.2) the following estimates hold

$$\text{(case I)}: \quad \frac{1}{2}(\lambda^{-1} - 1 - 2\epsilon)\|u\|_2^2 \le \mathcal{V} \le \frac{1}{2}(\lambda^{-1} + (1-\alpha)(1-\alpha+\epsilon))\|u\|_2^2,$$

$$\text{(case II)}: \quad \frac{1}{2}\left(\lambda^{-1} - \alpha - \frac{1}{2} - 2\epsilon\right)\|u\|_2^2 \le \mathcal{V} \le \frac{1}{2}(\lambda^{-1} + (1-\alpha)(1-\alpha+\epsilon))\|u\|_2^2,$$

$$\text{(case III)}: \quad \frac{1}{2}\left(\lambda^{-1} - \alpha - \frac{1}{2} - (\alpha+1)\epsilon\right)\|u\|_2^2 \le \mathcal{V} \le \frac{\lambda^{-1}}{2}\|u\|_2^2.$$

In the case of material damping we will assume that $(u, \dot{u}) \in \mathbf{H}_0^2[0,1] \times \mathbf{H}_0^2[0,1]$

We will not prove any result on the existence and uniqueness of solutions of system (2.13)–(2.17). We will investigate stability conditions only for solutions belonging to the space $\mathbf{H}_0^2[0,1] \times \tilde{\mathbf{L}}_2[0,1]$. Nevertheless, we have to point out that the existence of such solutions has not been strictly proved. Such a proof is based on the application of a semi-group technique which does not have any physical meaning, but requires a lot of purely mathematical technicalities.

Theorem 1. *If the functional of the reduced potential energy $\mathcal{V}$ is positive definite with respect to the norm $\|\cdot\|_2$, then the trivial solution of system* (2.13)–(2.17) $u(\sigma,\tau)=\dot{u}(\sigma,\tau)\equiv 0$ *is asymptotically stable with respect to the norm of the phase space* $\mathbf{H}_0^2[0,1]\times\tilde{\mathbf{L}}_2[0,1]$. *If the functional $\mathcal{V}$ admits negative values, then the trivial solution is unstable with respect to the above norms.*

Proof. Let us start with the case of instability. Many authors have developed techniques to prove Lyapunov stability or instability for systems with an infinite number of degrees of freedom [18–21]. For finite-dimensional systems the presence of dissipation is a simplifying factor. In this case one can simply apply the Barbashin-Krasovsky theorem to prove stability or instability. But for infinite-dimensional systems this theorem is not valid, as for example, is shown in [22]. To investigate systems with material damping, one needs to develop a more refined method which will be presented below.

To prove instability, we can take a modified variant of Chetaev's function [23]. Peculiarities of the use of Chetaev's method for infinite-dimensional cases are discussed in [18, 21, 24–26]. In a symbolical form that modified function can be written as follows

$$\mathcal{F}=\langle p,q\rangle+\Phi[q]$$

Here the expression $\langle p,\cdot\rangle$ should be understood as a linear functional on the space which the generalized coordinates q belong to. And the dissipative function Φ is calculated on generalized coordinates instead of generalized velocities.

Applying the above construction to the problem under consideration, we obtain

$$\mathcal{F}=\int_0^1\dot{u}u\,\mathrm{d}\sigma+\epsilon\dot{u}(1,\tau)u(1,\tau)+\frac{\lambda^{-1}\beta}{2}\int_0^1 u''^2\,\mathrm{d}\sigma \tag{3.5}$$

Using the Cauchy-Bunyakovskii inequality, we can obtain an upper estimate for the functional $\mathcal{F}$ absolute value

$$|\mathcal{F}|\le M(\|\dot{u}\|_*+\|u\|_2)\|u\|_2, \tag{3.6}$$

where M is a certain positive constant.

Hence, the functional $\mathcal{F}$ is bounded for any bounded solution of system (2.13)–(2.17).

Let $u(\sigma,\tau),\dot{u}(\sigma,\tau)$ be a solution of system (2.13)–(2.17) with a negative initial energy store, *i.e.*,

$$\mathcal{H}[\dot{u}(\cdot,0),u(\cdot,0)]=h<0.$$

Since the potential-energy functional admits negative values, such a solution really exists. Obviously, the following equality holds

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}\tau}=-\Phi.$$

This means that the function $\mathcal{H}[\dot{u}(\cdot,\tau),u(\cdot,\tau)]$ does not increase with respect to time τ. Using the system (2.13)–(2.17), let us derive a formula for the time derivative of the functional $\mathcal{F}$ calculated on the above solution.

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}\tau} &= \int_0^1 \dot{u}^2 \mathrm{d}\sigma + \epsilon \dot{u}^2(1,\tau) + \int_0^1 \ddot{u} u \mathrm{d}\sigma + \epsilon \ddot{u}(1,\tau) u(1,\tau) + \lambda^{-1}\beta \int_0^1 u'' \dot{u}'' \mathrm{d}\sigma \\
&= \int_0^1 \dot{u}^2 \mathrm{d}\sigma + \epsilon \dot{u}^2(1,\tau) + \lambda^{-1}\beta \int_0^1 u'' \dot{u}'' \mathrm{d}\sigma + \left[-\frac{1}{2} \int_0^1 (f(\alpha,\sigma) u')' u \mathrm{d}\sigma + \int_0^1 u^2 \mathrm{d}\sigma \right. \\
&\quad \left. -\lambda^{-1} \int_0^1 (u^{\mathrm{IV}} + \beta \dot{u}^{\mathrm{IV}}) u \mathrm{d}\sigma - \epsilon(\alpha - 1) \int_0^1 u'' u \mathrm{d}\sigma \right] \\
&\quad + [\epsilon u^2(1,\tau) + \epsilon(\alpha-1) u'(1,\tau) u(1,\tau) + \lambda^{-1}(u'''(1,\tau) + \beta \dot{u}'''(1,\tau)) u(1,\tau)].
\end{aligned}
$$

After some quite awkward calculations, including integration by parts, we arrive at the following formula

$$
\begin{aligned}
\dot{\mathcal{F}} &= \int_0^1 \dot{u}^2 \mathrm{d}\sigma + \epsilon \dot{u}^2(1,t) - \int_0^1 \left(\lambda^{-1} u''^2 - \frac{1}{2}(\alpha,\sigma) u'^2 - u^2 - \epsilon(\alpha-1) u'^2 \right) \mathrm{d}\sigma \\
&\quad -\epsilon u^2(1,\tau) = 2\mathcal{T} - 2\mathcal{V} = 2\mathcal{L}
\end{aligned} \tag{3.7}
$$

This means in particular that $\dot{\mathcal{F}} \geq -2\mathcal{V}$. As has already been noted, $\mathcal{H}[\dot{u}(\cdot,\tau), u(\cdot,\tau)]$ does not increase. Therefore, $\mathcal{V} = \mathcal{H} - \mathcal{T} \leq \mathcal{H} \leq h$. Consequently,

$$
\dot{\mathcal{F}} \geq -2h,
$$

which immediately leads to the following inequality

$$
\mathcal{F} \geq \mathcal{F}_0 - 2h\tau, \tag{3.8}
$$

where $\mathcal{F}_0$ is the initial value of the functional $\mathcal{F}$ and $\mathcal{F} \to +\infty$ as $\tau \to +\infty$.

If the equilibrium position under consideration were stable, the functional $\mathcal{F}$ would be bounded. So we arrive at a contradiction.

Let us now pass to the case of stability. If $\mathcal{V}$ is positive definite with respect to the norm $\| \cdot \|_2$, then the total energy functional $\mathcal{H}$ is positive definite with respect to the norm of the phase space $\mathbf{H}_0^2[0,1] \times \tilde{\mathbf{L}}_2[0,1]$. That means that there exist two positive constants c_1 and C_1 such that for any $u, \dot{u}$ the following inequality holds

$$
c_1 N \leq \mathcal{H} \leq C_1 N, \tag{3.9}
$$

where

$$
N = \|u\|_2^2 + \|\dot{u}\|_*^2.
$$

Let us consider the following Lyapunov function

$$
\mathcal{W} = \mathcal{H} + \mu \mathcal{F}, \tag{3.10}
$$

where μ is a certain positive small parameter.

Since $\mathcal{F}$ is a bounded quadratic form, this parameter can be choosen so that the functional $\mathcal{W}$ will be positive definite as well. Hence, there exist positive numbers c_2 and C_2 providing us with the estimate

$$
c_2 N \leq \mathcal{W} \leq C_2 N. \tag{3.11}
$$

From the previous calculations we have

$$\dot{\mathcal{W}} = \dot{\mathcal{H}} + \mu \dot{\mathcal{F}} = -\Phi + 2\mu \mathcal{L}. \tag{3.12}$$

According to Lemma 1,

$$\Phi \geq \frac{\lambda^{-1}\beta}{2} C^2 \|\dot{u}\|_*^2.$$

Hence, if μ is small enough

$$-\Phi + 2\mu \mathcal{T} \leq -c_3 \|\dot{u}\|_*^2,$$

where c_3 is a positive constant.

But in this case there exists another positive constant c_4 (possibly a very small one) such that

$$\dot{\mathcal{W}} \leq -c_4 N. \tag{3.13}$$

It follows from the estimate (3.11) that $N \geq C_2^{-1}\mathcal{W}$. Therefore, from the last inequality and (3.13), we obtain

$$\dot{\mathcal{W}} \leq -\varkappa \mathcal{W}, \tag{3.14}$$

where $\varkappa = c_4 C_2^{-1}$.

The inequality (3.14) means that the functional $\dot{\mathcal{W}}$ is negative definite, but not in a usual finite-dimensional sense, because an analogous lower estimate is not valid. Nevertheless, by integrating inequality (3.14) from 0 to τ, we obtain an exponential estimate for the Lyapunov function $\mathcal{W}$

$$\mathcal{W} \leq \mathcal{W}_0 \, \mathrm{e}^{-\varkappa\tau}. \tag{3.15}$$

Taking into account that $N \leq c_2^{-1}\mathcal{W}$ and using the inequality (3.15), we obtain the final estimate

$$N \leq c_2^{-1} \mathcal{W}_0 \, \mathrm{e}^{-\varkappa\tau}, \tag{3.16}$$

which is equivalent to the exponential asymptotic stability of the trivial relative equilibrium position.

If we have no material damping ($\beta = 0$), then we can use the total energy functional $\mathcal{H}$ as Lyapunov's function and prove the Lyapunov stability (not asymptotic!).

4. Influence of parameters

It is worth noting that it follows from the lower estimates for the functional $\mathcal{V}$ (see the previous section), that for any fixed values of the parameters α and ϵ we can choose λ so small that $\mathcal{V}$ will be positive definite. From a physical point of view this means, for instance, that a sufficiently stiff beam is stable. Precisely speaking, the following statement can be made:

Theorem 2. *For any fixed values of parameters α, ϵ there exists a positive number $\lambda^*(\alpha, \epsilon)$ such that for any $\lambda < \lambda^*$ the functional of the reduced potential energy $\mathcal{V}$ is positive definite.*

Proof. The idea of the proof is very simple. Using the lower estimates depending on values α for the functional $\mathcal{V}$ we obtained in the previous section (cases I-III), we can construct the desired 'function' $\lambda^*(\alpha, \epsilon)$ explicitly. Indeed, let it have the following form

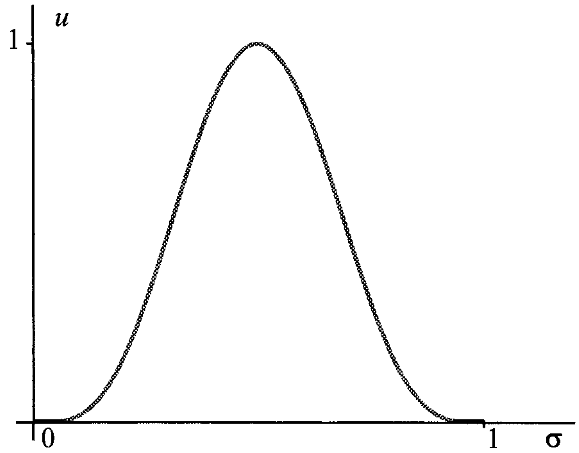

Figure 2. Graph of the function $u_*(\sigma)$ for the case $\alpha \geq 1$

$$\lambda^*(\alpha, \epsilon) = \begin{cases} (1+2\epsilon)^{-1}, & \text{Case I,} \\ (\alpha + 1/2 + 2\epsilon)^{-1}, & \text{Case II,} \\ (\alpha + 1/2 + (\alpha+1)\epsilon)^{-1}, & \text{Case III.} \end{cases} \tag{4.1}$$

Then it follows from the estimates mentioned that if $\lambda < \lambda^*$, the functional $\mathcal{V}$ is positive definite.

As has just been proved, for any values of α and ϵ the equilibrium position remains stable for sufficiently small λ, *i.e.*, if the beam is sufficiently stiff or if the value of the angular velocity is low enough. Let us consider an opposite situation. What happens if λ is fairly large? Precisely speaking, we will try to prove that for any fixed α and ϵ it is possible to find a number λ_*, such that for all $\lambda > \lambda_*$ the functional of the reduced potential energy $\mathcal{V}$ admits negative values. According to Theorem 1, this means instability. Let us formulate the above statement in the form of a theorem.

Theorem 3. *For any fixed values of the parameters α, ϵ there exists a $\mathbf{H}_0^2[0, 1]$ - function $u_*(\sigma)$ satisfying additionally the following boundary conditions*

$$u_*(0) = u_*'(0) = u_*''(1) = \epsilon(u_*(1) + (\alpha - 1)u_*'(1)) + \lambda^{-1}u_*'''(1) = 0, \tag{4.2}$$

such that $\mathcal{V}[u_*] < 0$ for any $\lambda > \lambda_*$ where $\lambda_*(\alpha, \epsilon)$ is a certain positive number.

Proof. The desired function $u_*(\sigma)$ is different for different values of α. If $\alpha \geq 1$, we can construct u_* as follows

$$u_*(\sigma) = 1 + \sum_{n=1}^{4} a_{2n}(\sigma - 1/2)^{2n}. \tag{4.3}$$

This function is mirror-symmetric with respect to a vertical line intersecting the axis $O\sigma$ at the point $\sigma = 1/2$ (see Figure 2). We can choose the coefficients a_{2n}, $n = 1, 2, 3, 4$ so that

$$u_*(0) = u_*'(0) = u_*''(0) = u_*'''(0) = 0.$$

By solving a simple system of linear algebraic equations, we can find that

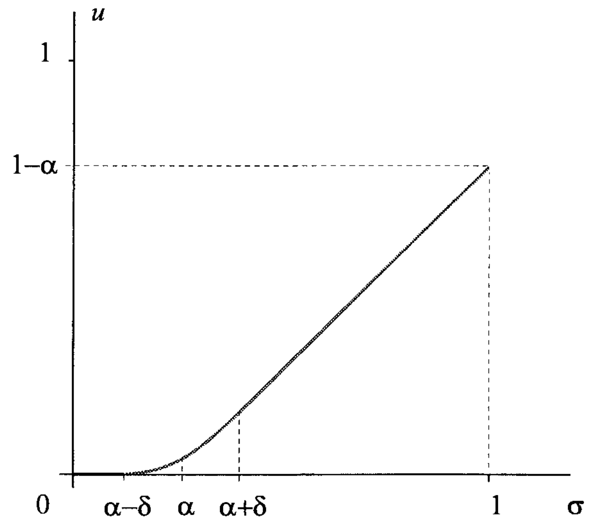

Figure 3. Graph of the function $u_*(\sigma)$ for the case $\alpha < 1$

$$a_2 = -16, \quad a_4 = 96, \quad a_6 = -256, \quad a_8 = 256$$

Since $u_*(\sigma) = u_*(1-\sigma)$,

$$u_*(1) = u'_*(2) = u''_*(1) = u'''_*(1) = 0$$

and the boundary conditions (4.2) are satisfied automatically.

After some simple but awkward calculations we obtain

$$\mathcal{V}[u_*] = -\frac{16384}{765765}\left(-\frac{7344}{\lambda} + 68(\alpha-1)(1+2\epsilon) + 27\right). \tag{4.4}$$

Since $\alpha \geq 1$, the above expression is negative for $\lambda = +\infty$. Therefore, it remains negative for sufficiently large values of λ.

The case $\alpha < 1$ requires another construction. Let

$$u_*(\sigma) = \begin{cases} 0, & 0 \leq \sigma < \alpha - \delta, \\ U_*(\sigma), & \alpha - \delta \leq \sigma < \alpha + \delta, \\ \sigma - \alpha, & \alpha + \delta \leq \sigma \leq 1, \end{cases} \tag{4.5}$$

(see Figure 3), where

$$U_*(\sigma) = \sum_{n=4}^{7} b_n(\sigma - \alpha + \delta)^n \tag{4.6}$$

and δ is a small positive number.

It is not difficult to note that the boundary conditions (4.2) are satisfied if the coefficients $b_n, n = 4, 5, 6, 7$, are chosen so that the whole function $u_*(\sigma)$ is smooth enough. We can choose those coefficients so that $u_* \in \mathbb{C}^3[0, 1]$. Indeed, the conditions

$$U_*(\alpha - \delta) = U'_*(\alpha - \delta) = U''_*(\alpha - \delta) = U'''_*(\alpha - \delta) = 0$$

are satisfied automatically.

So, we need to satisfy the following three conditions

$$U_*(\alpha+\delta)=\delta, \quad U'_*(\alpha+\delta)=1, \quad U''_*(\alpha+\delta)=0, \quad U'''_*(\alpha+\delta)=0.$$

Solving an algebraic system for b_n, $n=4,5,6,7$, we obtain

$$b_4=\frac{5}{16}\delta^{-3}, b_5=-\frac{3}{16}\delta^{-4}, \quad b_6=\frac{1}{32}\delta^{-5}, \quad b_7=0.$$

Now we can calculate the value of the functional $\mathcal{V}$ on the constructed form $u_*(\sigma)$ and get the following expression

$$\mathcal{V}[u_*]=\frac{5}{14\delta\lambda}-\frac{25}{462}(\alpha-1)(\alpha-1-2\epsilon)\delta-\frac{25}{6006}\delta^3. \tag{4.7}$$

Since $\alpha<1$, the above expression is negative for $\lambda=+\infty$ and any δ and ϵ. Hence, the functional $\mathcal{V}$ remains negative for sufficiently large λ.

Here we need to discuss the role of the attached point mass. We can see from the expression for the potential energy (3.2) that, if $\alpha>1$, the functional $\mathcal{V}_1$ is negative and the point mass plays a destabilizing role. On the other hand, if $\alpha<1$, that role is unclear. From a physical standpoint, the centrifugal forces acting upon the mass try to bend the beam if its length l does not exceed the radius R of the wheel. Vice versa, those forces try to stretch the beam if $l>R$. It may seem that one can stabilize the rectilinear shape of the beam by attaching an essentially large point mass for any values of the angular velocity. The previous statement shows that this is impossible. For any mass m we can choose an angular velocity ω, so that the rectilinear form of the beam is unstable.

5. Stability threshold

Let us discuss the structure of the subsets in the 4D space of real parameters $(\lambda,\alpha,\epsilon,\beta)$ for which the rectilinear shape of the beam is asymptotically stable or unstable respectively. Those two sets are divided by a 3D hypersurface Σ^3, the so-called stability threshold. Let us first show that the dissipation parameter β does not influence the stability threshold Σ^3 in the following sense. Let $\beta=0$ and Σ^2 be a 2D surface dividing domains of stability and instability in the 3D space of parameters $(\lambda,\alpha,\epsilon)$. Then $\Sigma^3=\Sigma^2\times\{\beta>0\}$. Indeed, as was shown above, the rectilinear shape is stable if $\mathcal{V}$ is positive definite and unstable if $\mathcal{V}$ admits negative values independently of the value of the parameter β. Hence, Σ^3 has the above form. Further, let us show that

$$\Sigma^2=\{(\lambda,\alpha,\epsilon):\lambda=\lambda_1(\alpha,\epsilon)\}$$

where λ_1 is a certain function of α,ϵ which, for simplicity, will be also called ‘the stability threshold’.

Let us fix certain values of α and ϵ. Since $\mathcal{V}$ is positive definite if $\lambda<\lambda^*(\alpha,\epsilon)$ and admits negative values if $\lambda>\lambda_*(\alpha,\epsilon)$, there is a $\lambda_1(\alpha,\epsilon)$, $\lambda^*\le\lambda_1\le\lambda_*$

$$\lambda_1=\inf\{\lambda:\mathcal{V}[u]\ge 0,\ u\in\mathbf{H}_0^2[0,1]\} \tag{5.1}$$

Therefore, if $\lambda>\lambda_1$, the functional $\mathcal{V}$ admits negative values, which means instability. Let $\lambda<\lambda_1$ and $\lambda_1-\lambda=\Delta$. Then it is easy to show that

$$\mathcal{V} \geq \frac{\Delta}{2\lambda\lambda_1}\|u\|_2^2 \quad \text{for any } u \in \mathbf{H}_0^2[0,1]$$

which means that $\mathcal{V}$ is positive definite and the rectilinear shape of the beam is stable.

The desired λ_1 is obviously the minimal positive λ for which there is a smooth nonzero function $u_1(\sigma)$ such that $\mathcal{V}[u_1] = 0$. Due to standard ideas of variational calculus [27], the function u_1 is a minimum of $\mathcal{V}$ and λ_1, u_1, is a solution of the following Sturm-Liouville problem of fourth order

$$u^{\mathrm{IV}} + \lambda\left(\frac{1}{2}(f(\alpha,\sigma)u')' - u + \epsilon(\alpha-1)u''\right) = 0, \tag{5.2}$$

$$u(0) = u'(0) = 0, \tag{5.3}$$

$$u''(1) = u'''(1) + \lambda\epsilon(u(1) + (\alpha-1)u'(1)) = 0, \tag{5.4}$$

Let us show that the spectrum of the above problem is discrete and real, consists of only eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ and ∞ is the only limiting point of it. Indeed, by integrating (5.2) four times and taking into account the boundary conditions (5.3) and (5.4), instead of the Sturm-Liouville problem (5.2), (5.3), (5.4), we can obtain the following integral equation

$$\begin{aligned} \mu u(\sigma) \ &= g_1(\sigma,\alpha,\epsilon)u(1) + \int_0^{\sigma}\int_0^{\xi}\Big\{g_2(\eta,\alpha,\epsilon)u(\eta) + \int_{\eta}^{1}\Big[g_3(\zeta,\alpha)u(\zeta) \\ &\quad + \int_{\zeta}^{1} u(\vartheta)\mathrm{d}\vartheta\Big]\mathrm{d}\zeta\Big\}\,\mathrm{d}\eta\mathrm{d}\xi, \end{aligned} \tag{5.5}$$

which can be symbolically represented in an operator form

$$\mu u = Tu.$$

Here $\mu = \lambda^{-1}$ and $g_1(\sigma,\alpha,\epsilon) = -\frac{\epsilon}{6}\sigma^2(\sigma-\alpha)$, $g_2(\sigma,\alpha,\epsilon) = -\frac{1}{2}((\sigma-\alpha)^2 - (1-\alpha)^2 - 2\epsilon(1-\alpha))$, $g_3(\sigma,\alpha) = \sigma - \alpha$.

The operator T maps the space $\mathbf{H}_0^2[0,1]$ into itself. To prove that it is self-adjoint, we need to show that for any two functions $u_{(1)}, u_{(2)}$ belonging to $\mathbf{H}_0^2[0,1]$

$$\langle Tu_{(1)}, u_{(2)}\rangle_2 = \langle u_{(1)}, Tu_{(2)}\rangle_2.$$

After some calculations we obtain

$$\begin{aligned} \langle Tu_{(1)}, u_{(2)}\rangle_2 \ &= \int_0^1 (Tu_{(1)})''(\sigma)u''_{(2)}(\sigma)\mathrm{d}\sigma \\ &= \epsilon u_{(1)}(1)u_{(2)}(1) + \int_0^1 u_{(1)}(\sigma)u_{(2)}(\sigma)\mathrm{d}\sigma = \langle u_{(1)}, Tu_{(2)}\rangle_2. \end{aligned} \tag{5.6}$$

By using the Rellich criterium on the compactness of sets of functions in $\mathbf{L}_2$-spaces [28], we can prove that T is a compact linear operator in $\mathbf{H}_0^2[0,1]$. Since T is a compact self-adjoint operator, its spectrum $\Sigma(T)$ is real, bounded and discrete and consists of only eigenvalues of T $\{\mu_j\}_{j=1}^{\infty}$ and the point $\mu = 0$. That point is the only limiting point of $\Sigma(T)$. After rewriting those properties in terms of λ, we obtain the desired properties of the spectrum of the Sturm-Liouville problem (5.2)–(5.4).

Even if $\epsilon = 0$, *i.e.*, we have no attached mass, the Sturm-Liouville problem (5.2), (5.3), (5.4) is not integrable, and to get a reasonable approximate solution, one needs to apply a kind of perturbation technique (see [12]). Nevertheless, it is possible to simplify (5.2), (5.3), (5.4) by reducing it to another eigenvalue problem of second order. First, let us note that $u(\sigma) = \sigma - \alpha$ is a particular solution of the differential equation of fourth order (5.2). Then it can be reduced to a differential equation of third order by substituting $u(\sigma) = (\sigma - \alpha)v'(\sigma)$. The necessary calculations yield the following differential equation in terms of $v(\sigma)$

$$(\sigma-\alpha)v'''+4v''+\lambda\left(\frac{1}{2}[(\sigma-\alpha)^2-(1-\alpha)^2-2\epsilon(1-\alpha)]((\sigma-\alpha)v'+2v)+(\sigma-\alpha)^2v\right)=0. \tag{5.7}$$

The first boundary condition in (5.3) disappears, and the other one and (5.4) take the form

$$v(0) = (1-\alpha)v'(1) + 2v(1) = (1-\alpha)v''(1) + 3v'(1) - \lambda\epsilon(1-\alpha)^2 v(1) = 0. \tag{5.8}$$

By multiplying Equation (5.7) by $\sigma - \alpha$ and combining terms in a proper way, we arrive at the following equation

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\{(\sigma-\alpha)^2v''+2(\sigma-\alpha)v'-2v\}+\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}\sigma}\{(\sigma-\alpha)^2[(\sigma-\alpha)^2-(1-\alpha)(1-\alpha+2\epsilon)]v\}=0.$$

Consequently,

$$\{(\sigma-\alpha)^2v''+2(\sigma-\alpha)v'-2v\}+\frac{\lambda}{2}\{(\sigma-\alpha)^2[(\sigma-\alpha)^2-(1-\alpha)(1-\alpha+2\epsilon)]v\}=C, \tag{5.9}$$

where C is a real constant.

To calculate that constant C, we substitute $\sigma = 1$ in (5.9). Using the boundary conditions (5.8), we can easily prove that $C = 0$. This yields a differential equation of second order with boundary conditions

$$v(0) = (1-\alpha)v(1) + 2v(1) = 0.$$

The differential equation obtained can be brought into a simpler form by means of the following change of variables

$$z = \sigma - \alpha, \quad v(\sigma) = z^{-2}w(z), \quad \gamma = 1 - \alpha.$$

The desired Sturm-Liouville problem reads

$$w'' - \frac{2}{z}w' + \frac{\lambda}{2}(z^2 - \gamma^2 - 2\epsilon\gamma)w = 0, \tag{5.10}$$

$$w(\gamma - 1) = w'(\gamma) = 0. \tag{5.11}$$

All the above transformations excite no questions about their validity if $\alpha > 1$ (the beam is longer than the radius of the wheel). If $\alpha < 1$, the solutions of Equation (5.10) may have singularities at $z = 0$. In particular, that method does not work if $\alpha = 1$ because of the loss of a boundary condition.

There are several papers [1, 10, 11, 12] where explicit formulae are given for the stability threshold in the absence of the point mass. All those formulae have the same imperfection. They work more or less properly only in the region of large α. Those formulae were obtained

by carrying out only a small number of steps of some recurrent methods. That is why their accuracy is far from being perfect.
Now we derive another explicit formula for the stability threshold using the reduced Sturm-Liouville problem (5.10), (5.11).

As follows from the analytical theory of linear differential equations [29], we can search for solutions of Equation (5.10) in the form of series

$$w(z) = z^{\rho} \sum_{k=0}^{\infty} w_k(\log\, z) z^k, \tag{5.12}$$

where w_k are certain polynomials.

It is not difficult to show that Equation (5.10) possesses two independent particular solutions, an even one with $\rho_1 = 0$ and an odd one with $\rho_2 = 3$, which are analytic in a neighborhood of the singularity $z = 0$. Indeed, let us substitute

$$w_1(z) = \sum_{k=0}^{\infty} a_k z^k, \quad \text{where } a_0 = 1 \tag{5.13}$$

in Equation (5.10).

By collecting terms at z^{-1}, z^0, z, z^2 we easily calculate $a_1 = 0$, $a_2 = -\frac{\lambda}{4}\gamma(\gamma + 2\epsilon)$ and a_3 can be chosen arbitrarily, so we can put $a_3 = 0$.

Higher-order coefficients can be calculated recurrently by equating terms at z^{k-2}.

$$(k^2 - 3k)a_k + \frac{\lambda}{2}(a_{k-4} - \gamma(\gamma + 2\epsilon))a_{k-2} = 0, \quad k = 4, 5, \ldots . \tag{5.14}$$

Subsequently solving the chain of Equations (5.14), we can prove that all the coefficients with even numbers are equal to zero.

Let us now substitute

$$w_2(z) = z^3 \sum_{k=0}^{\infty} b_k z^k, \quad \text{where } b_0 = 1 \tag{5.15}$$

in Equation (5.10). After equating terms at z^2, z^3, z^4, we obtain $b_1 = 0$, $b_2 = \frac{\lambda}{20}\gamma(\gamma + 2\epsilon)$, $b_3 = 0$.

Further, we can obtain the following chain of equations

$$(k^2 + 3k)b_k + \frac{\lambda}{2}(b_{k-4} - \gamma(\gamma + 2\epsilon))b_{k-2} = 0, \quad k = 4, 5, \ldots \tag{5.16}$$

and easily prove that all b_k with even numbers are equal to zero.

Hence, we obtain an asymptotic representation of the fundamental system of solutions of (5.10) near the singularity $z = 0$.

$$w_1(z) = 1 - \frac{\lambda}{4}\gamma(\gamma + 2\epsilon)z^2 + O(z^4),\, w_2(z) = z^3 + \frac{\lambda}{20}\gamma(\gamma + 2\epsilon)z^5 + O(z^7). \tag{5.17}$$

To get an equation connecting parameters $\lambda, \gamma, \epsilon$, we should consider the equation

$$\begin{vmatrix} w_1(\gamma - 1) & w_2(\gamma - 1) \\ w_1'(\gamma) & w_2'(\gamma) \end{vmatrix} = 0. \tag{5.18}$$

Keeping only two coefficients in the expansion of the solutions $w_1(z)$, $w_2(z)$ in (5.17), we obtain a quadratic equation for λ from Equation (5.18). If $\lambda > 1$, then the singularity $z = 0$ lies outside the segment $[\gamma - 1, \gamma]$ and the above method seems to be inva1id. Moreover, as has been already noted, the Sturm-Liouville problem (5.10), (5.11) is not equivalent to the initial one if $\alpha = 1$. So, we may hope that we obtain a plausible formula only in the neighbourhood of $\alpha = 0$ ($\gamma = 1$). After solving the quadratic equation mentioned and taking a root positive for $0 < \gamma < 1$, we obtain

$$\lambda_1 = 2\frac{-10 + 15\gamma + \sqrt{20 - 36\gamma - 75\gamma^2 + 240\gamma^3 - 180\gamma^4 + 36\gamma^6}\sqrt{5}}{\gamma(\gamma + 2\epsilon)(3\gamma^3 + 6\gamma^2 - 6\gamma + 2)(\gamma - 1)^2}. \tag{5.19}$$

By using formula (5.19) for the stability threshold λ_1, we can obtain an asymptotic formula for it as $\alpha \to 0+$

$$\lambda_1 \sim \frac{4}{1 + 2\epsilon}\alpha^{-2}. \tag{5.20}$$

We will show that the order of the asymptotics of the stability threshold is found correctly while the corresponding coefficient is two times greater. Hence, formula (5.19) for the stability threshold should be applied very carefully even for small α.

Through the method suggested in [12], we obtain now a precise asymptotics for λ_1 as $\alpha \to 0+$. By doing the following change

$$\lambda = \frac{2\nu}{\alpha^2}, \quad z = \alpha\zeta,$$

we obtain a new eigenvalue problem

$$w''(\zeta) - \frac{2}{\zeta}w'(\zeta) + \nu(\alpha^2 z^2 - \gamma^2 - 2\epsilon\gamma)w(\zeta) = 0, \tag{5.21}$$

$$w(-1) = w'(\gamma/\alpha) = 0. \tag{5.22}$$

Let us pass to the limit as $\alpha \to 0+$ in (5.21), (5.22) bearing in mind that $\alpha + \gamma = 1$. After that (5.21), (5.22) become

$$w''(\zeta) - \frac{2}{\zeta}w'(\zeta) - \nu(1 + 2\epsilon)w(\zeta) = 0, \tag{5.23}$$

$$w(-1) = w'(+\infty) = 0. \tag{5.24}$$

The general solution of the differential equation (5.23) reads

$$w(\zeta) = C_1 w_1(\zeta) + C_2 w_2(\zeta),$$

where

$$w_1(\zeta) = \varkappa\zeta \cosh \varkappa\zeta - \sinh \varkappa\zeta, w_1(\zeta) = \varkappa\zeta \sinh \varkappa\zeta - \cosh \varkappa\zeta,$$

and $\varkappa = \sqrt{\nu(1 + 2\epsilon)}$.

Since both solutions $w_1(\zeta)$, $w_2(\zeta)$ and their derivatives go to $+\infty$ as $\zeta \to +\infty$, to satisfy the second boundary condition, we have to put $C_1 = -C_2$. Then the second boundary condition gives us an equation

$$(\varkappa - 1)e^{\varkappa} = 0,$$

which has the only real positive solution $\varkappa = 1$.

Going back to the original variables, we obtain the necessary asymptotics

$$\lambda_1 \sim \frac{2}{1+2\epsilon}\alpha^{-2}, \tag{5.25}$$

as $\alpha \to 0+$ which coincides with the asymptotics obtained in [12] at the point $\epsilon = 0$.

Let us find now the asymptotics of the stability threshold λ_1 as $\alpha \to +\infty$. To do so, let us carry out the change of variables

$$\lambda = \frac{\Omega^2}{\alpha}, \quad z = \zeta - \alpha,$$

which brings the eigenvalue problem (5.10), (5.11) into the form

$$w''(\zeta) - \frac{2}{\zeta - \alpha}w'(\zeta) + \frac{\Omega^2}{2\alpha}(\zeta^2 - 2\epsilon - 1 - 2\alpha(\zeta - \epsilon - 1))w(\zeta) = 0, \tag{5.26}$$

$$w(0) = w'(1) = 0. \tag{5.27}$$

After passing to the limit as $\alpha \to +\infty$ and carrying out the change of the independent variable $x = \Omega(1 + \epsilon - \zeta)$, we arrive at the following boundary-value problem

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}(x) + xw(x) = 0, \tag{5.28}$$

$$w(\Omega(1+\epsilon)) = \frac{\mathrm{d}w}{\mathrm{d}x}(\Omega\epsilon) = 0. \tag{5.29}$$

Equation (5.28) can be transformed to Bessel's equation of order 1/3 [29]

$$X^2\frac{\mathrm{d}^2 W}{\mathrm{d}X^2}(X) + X\frac{\mathrm{d}W}{\mathrm{d}X}(X) + \left(X^2 - \left(\frac{1}{3}\right)^2\right)W(X) = 0 \tag{5.30}$$

by means of the change of variables

$$w(x) = \sqrt{x}W(X), \quad X = \frac{2}{3}\sqrt{x^3}.$$

Solutions of (5.30) are the so-called cylindric functions $Z_p(X)$ which can be written as

$$Z_p(X) = C_1 J_p(X) + C_2 J_{-p}(X)$$

for the non-integer $p = 1/3$, where J_p, J_{-p} are Bessel functions of order 1/3 and $-1/3$ respectively [29].

The first boundary condition from (5.29) gives us the following equation

$$\sqrt{\Omega(1+\epsilon)}Z_p\left(\frac{2}{3}\sqrt{\Omega^3(1+\epsilon)^3}\right) = 0. \tag{5.31}$$

The second boundary condition provides us with another equation connecting parameters Ω and ϵ

$$\frac{1}{2\sqrt{\Omega\epsilon}} Z_p\left(\frac{2}{3}\sqrt{\Omega^3\epsilon^3}\right) + \Omega\epsilon \frac{\mathrm{d}Z_p}{\mathrm{d}X}\left(\frac{2}{3}\sqrt{\Omega^3\epsilon^3}\right) = 0. \tag{5.32}$$

Let us rewrite Equation (5.32) in a more appropriate form by designating

$$a = \frac{2}{3}\sqrt{\Omega^3\epsilon^3}$$

Then (5.32) reads

$$\frac{1}{2}\left(\frac{2}{3a}\right)^{1/3}\left(Z_p(a) + 3aZ'_p(a)\right) = 0. \tag{5.33}$$

To proceed further, we need the following identity [29]

$$(a^p Z_p(a))' = a^p Z_{p-1}(a).$$

On the other hand,

$$(a^p Z_p(a))' = a^p Z'_p(a) + pa^{p-1} Z_p(a).$$

Hence,

$$aZ'_p(a) = -pZ_p(a) + aZ_{p-1}(a).$$

Obviously, $p - 1 = -2/3$ but for non-integer q $Z_q(X) \equiv Z_{-q}(X)$ and Equation (5.33) for nonzero a can be rewritten as

$$\left(\frac{3a}{2}\right)^{2/3} Z_{2p}(a) = 0 \tag{5.34}$$

or

$$Z_{2/3}\left(\frac{2}{3}\sqrt{\Omega^3\epsilon^3}\right) = 0. \tag{5.35}$$

Equations (5.31) and (5.35) can be treated as a linear system for the arbitrary constants C_1, C_2. The condition of its nontrivial solvability looks as follows

$$\begin{aligned} F(\Omega,\epsilon) = {} & J_{1/3}\left(\frac{2}{3}\sqrt{\Omega^3(1+\epsilon)^3}\right) J_{-2/3}\left(\frac{2}{3}\sqrt{\Omega^3\epsilon^3}\right) \\ & - J_{-1/3}\left(\frac{2}{3}\sqrt{\Omega^3(1+\epsilon)^3}\right) J_{2/3}\left(\frac{2}{3}\sqrt{\Omega^3\epsilon^3}\right) = 0. \end{aligned} \tag{5.36}$$

This is a transcendent equation connecting parameters Ω and ϵ. Let us assume that we managed to solve it somehow with respect to $\Omega, \Omega = \Omega_1(\epsilon)$ being the minimal positive root of (5.36). Then the desired asymptotic representation for the stability threshold λ_1 as $\alpha \to +\infty$ would have the following form

$$\lambda_1 \sim (\Omega_1(\epsilon))^2 \alpha^{-1}. \tag{5.37}$$

The above procedure is not valid in the absence of the point mass. Indeed, $a = 0$ if $\epsilon = 0$. Let us pass to the limit as $\to 0+$ in (5.34). Taking into account that [29]

$$J_q(X) \sim \frac{X^q}{2^q \Gamma(q+1)} \quad \text{as} \quad X \to 0+,$$

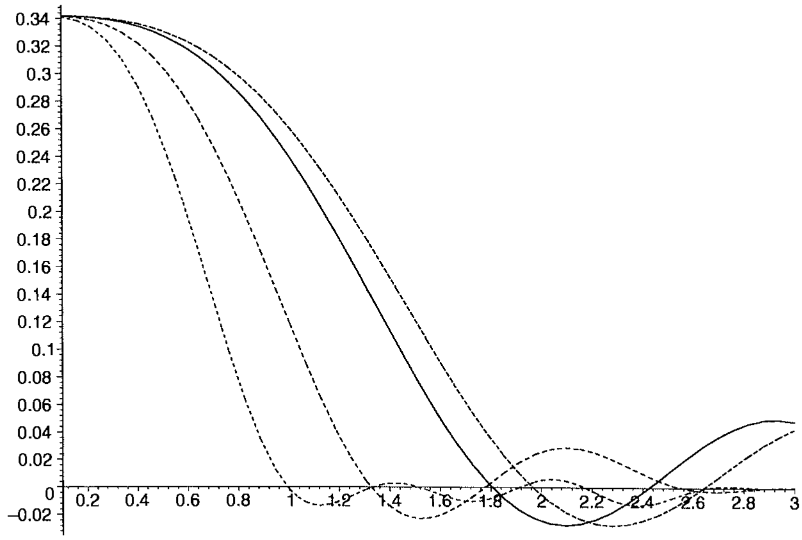

Figure 4. Graphs of the function $F(\Omega,\epsilon)$ for values of $\epsilon = 0{\cdot}01$, $0{\cdot}1$, $0{\cdot}5$,1

we obtain

$$\lim_{a\to 0+}\left(\frac{3a}{2}\right)^{2/3}\big(C_1J_{2/3}(a)+C_2J_{-2/3}(a)\big)=\frac{C_2 3^{2/3}}{\Gamma\left(\frac{1}{3}\right)}=0.$$

Consequently, $C_2 = 0$ and via the relation (5.31) we obtain the following equation

$$J_{1/3}\left(\frac{2}{3}\sqrt{\Omega^3}\right)=0. \tag{5.38}$$

The equation $J_{1/3}(X) = 0$ has an increasing sequence of positive real roots $\{j_{1/3}^{(k)}\}_{k=1}^{\infty}$. It is known that $j_{1/3}^{(1)} \approx 2{\cdot}902586248$ is the minimal root of the above equation. Then the stability threshold has the following asymptotics as $\alpha \to +\infty$

$$\lambda_1 \sim \left(\frac{3j_{1/3}^{(1)}}{2}\right)^{\frac{4}{3}}\alpha^{-1}. \tag{5.39}$$

If we substitute the approximate value of $j_{1/3}^{(1)}$, the asymptotic expression (5.39) becomes

$$\lambda_1 \sim 7{\cdot}109436669\alpha^{-1} \tag{5.40}$$

If we calculate the first asymptotic term for the stability threshold, using explicit formulae from [1, 10, 12], we obtain

$$\lambda_1 \sim 8\alpha^{-1} \tag{5.41}$$

for the formula from [1]

$$\lambda_1 \sim 7{\cdot}869663694\alpha^{-1} \tag{5.42}$$

for the formula from [10] and

$$\lambda_1 \sim 8{\cdot}291796068\alpha^{-1} \tag{5.43}$$

for the formula from [12].

As we can see, al1 the explicit formulae for the stability threshold which use an approximate analytic method give us an excessive estimate for large α.

6. Conclusions

We have investigated linear vibrations of a rotating elastic beam with an attached point mass. A partial differential equation of motion with the corresponding boundary conditions has been derived. We have investigated the stability of the trivial relative equilibrium position of the beam with respect to the norms of Sobolev-like functional spaces. We have shown that the equilibrium position under consideration is stable if and only if the functional of the reduced potential energy is positive definite. As the main tool, the Lyapunov direct method has been used. To prove instability, we have used a non-standard form of Chetayev's function. Since the famous Barbashin-Krasovsky theorem is not valid for mechanical system with an infinite number of degrees of freedom, we have had to use a refined version of the Lyapunov theorem on asymptotic stability and a certain perturbation of the total energy as a proper Lyapunov function. We have proved that the point mass cannot stabilize the rectilinear equilibrium position of the beam. Roughly speaking, that means that for any value of the attached mass, there exists a critical value of the angular velocity such that the equilibrium position is unstable if the angular velocity exceeds the above critical value. The main idea of the proof is to find a configuration of the beam for which the value of the reduced potential energy is negative. We have discussed the problem of deriving an explicit or asymptotic formula for the so-called stability threshold. First, the 'static' fourth-order boundary-value problem has been reduced to a second-order one with a singularity. By using that reduction and a power-series expansion for solutions, we have obtained an explicit (but approximate!) formula for the stability threshold. We have obtained some asymptotic formulae for the stability threshold both for small and large values of the parameter α. It turns out that the above explicit formula provides us with the correct order of asymptotics as $\alpha \to 0+$, while the corresponding coefficient is twice as large. That fact has been discovered by means of a perturbation technique developed earlier in [12]. That means that the above formula should be used very carefully. All the explicit formulae derived by authors of [1, 10, 12] suffer from analogous drawbacks. We have shown that for the case $\alpha \to +\infty$ a zero approximation can be treated as a parametric boundary-value problem for Bessel's equation of order 1/3. That allows us to obtain a transcendental algebraic equation for the coefficient at the first term of the corresponding asymptotics. We have compared the result obtained with results from [1, 10, 12]. It turns out that all the earlier results give an excessive estimate for the case of large α. But in that study we have assumed that the point mass and the mass of the beam are comparable. This means that the value of parameter ϵ is quite moderate. If it is large enough, some estimates obtained will no longer be valid and we have to apply other tools.

Appendix, Phase space

Let us describe the structure of the phase space of the system under consideration. Obviously, $(u, \dot{u}) \in \mathbf{H}_0^2[0,1] \times \tilde{\mathbf{L}}_2[0,1]$, where the Sobolev space $\mathbf{H}_0^2[0,1]$ is the closure of the space of $\mathbf{C}^\infty[0,1]$-functions $q(\sigma)$ satisfying boundary conditions $q(0) = q'(0) = 0$ with respect to the norm

$$\|q\|_2 = \sqrt{\int_0^1 q''^2 \mathrm{d}\sigma},$$

and $\tilde{\mathbf{L}}_2[0,1]$ is the closure of the space of $\mathbf{H}_0^2[0,1]$-functions $p(\sigma)$ with respect to the norm

$$\|p\|_* = \sqrt{p^2(1) + \int_0^1 p^2 \,\mathrm{d}\sigma}.$$

In the main text we also use the fact that $\mathbf{H}_0^2[0,1]$ is a Hilbert space with the following scalar product

$$\langle q_{(1)}, q_{(2)}\rangle_2 = \int_0^1 q''_{(1)}(\sigma) q''_{(2)}(\sigma)\mathrm{d}\sigma$$

for any two functions $q_{(1)}, q_{(2)} \in \mathbf{H}_0^2[0,1]$.

In the case of material damping we have to require slightly more regarding the smoothness of solutions of the system (2.13)–(2.17). More accurately, velocities $\dot{u}(\sigma,\tau)$ must at least belong to the functional space $\mathbf{H}_0^2[0,1]$ (see (3.4)).

Besides spaces $\mathbf{H}_0^2[0,1]$ and $\tilde{\mathbf{L}}_2[0,1]$ with norms $\|\cdot\|_2$ and $\|\cdot\|_*$, we will consider also the Sobolev space $\mathbf{H}_0^2[0,1]$ of functions $q(\sigma)$ satisfying the boundary condition $q(0) = 0$ and constructed in a way similar to the construction of $\mathbf{H}_0^2[0,1]$ with the norm

$$\|q\|_1 = \sqrt{\int_0^1 q'^2 \mathrm{d}\sigma},$$

and the space of quadratically integrable functions $\mathbf{L}_2[0,1]$ with the norm

$$\|q\|_0 = \sqrt{\int_0^1 q^2 \mathrm{d}\sigma},$$

From a formal point of view, a point mass is separated from the system as a whole and provides us with an additional degree of freedom. Moreover, if we look at the dissipative function (3.4), the dissipation may seem to be incomplete. Nevertheless, we are able to prove the following statement:

Lemma 1. *There exists a positive constant C such that for any function* $p(\sigma)$ *from* $\mathbf{H}_0^2[0,1]$ *the following inequality holds*

$$\|p\|_2 \geq C\|p\|_*. \tag{A.1}$$

Proof. This statement is based on two lemmas which are in a sense variants of Sobolev's embedding theorems [30] in the one-dimensional case.

Lemma 2. *Any function* $q \in \mathbf{H}_0^1[0,1]$ *satisfies the following inequality*

$$\|q\|_1 \geq \|q\|_0. \tag{A.2}$$

Proof. If we manage to prove this lemma for an arbitrary function $q(\sigma)$ from $\mathbf{C}^\infty[0,1]$ such that $q(0) = 0$, then the same statement is valid for any $q \in \mathbf{H}_0^1[0,1]$. Indeed, let us consider a functional sequence from $\mathbf{C}^\infty[0,1]$ approximating the function $q(\sigma)$. If the desired inequality is valid for any representative of the approximating sequence, it is also valid for the limiting function $q(\sigma)$. Let us denote $\|q\|_0$ as a. Since $q(0) = 0$, then, by applying a procedure of integrating by parts, we obtain

$$a^2 = -\int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}\sigma}\int_s^1 q(\xi)\mathrm{d}\xi\right) q(\sigma)\mathrm{d}\sigma = \int_0^1 q'(\sigma)\left(\int_\sigma^1 q(\xi)\mathrm{d}\xi\right)\mathrm{d}\sigma.$$

Further, the application of the Cauchy-Bunyakovskii inequality results in the following estimate

$$a^2 \le \|q\|_1 \sqrt{\int_0^1 \left(\int_\sigma^1 q(\xi)\mathrm{d}\xi\right)^2 \mathrm{d}\sigma}.$$

On the other hand,

$$\left|\int_\sigma^1 q(\xi)\mathrm{d}\xi\right|^2 \le \left(\int_\sigma^1 1\cdot|q|\mathrm{d}\xi\right)^2 \le \sigma \int_\sigma^1 q^2(\xi)\mathrm{d}\xi \le a^2.$$

Then the following inequality

$$a^2 \le a\|q\|_1$$

closes the proof of the lemma.

Lemma 3. *Any function* $q \in \mathbf{H}_0^1[0,1]$ *satisfies the following inequality*

$$\|q\|_1 \ge C\|q\|_* \tag{A.3}$$

for a certain positive constant C.

Proof. Let us consider the following functional on $\mathbf{H}_0^1[0,1]$

$$\mathcal{I} = \frac{\|q\|_1^2}{\|q\|_*^2}. \tag{A.4}$$

Functional (A.4) is positive. Let us find $C^2 = \inf \mathcal{I}[q]$, $q \in \mathbf{H}_0^2[0,1]$. According to the rules of variational calculus [27], C^2 is the lowest positive eigenvalue of the following Sturm-Liouville problem

$$q''(\sigma) + \lambda q(\sigma) = 0, \quad q(0) = 0, \quad q'(1) - \lambda q(1) = 0. \tag{A.5}$$

It is quite easy to find solutions of problem (A.5): $q(\sigma) = A \sin\sqrt{\lambda}\sigma$, $\sqrt{\lambda} = \cot\sqrt{\lambda}$. Then C will be the lowest root of the last equation, $C \approx 0{\cdot}86033$. Taking this value for C, we obtain inequality (A.3).

Now let us complete the proof of Lemma 1. By combining Lemmas 2 and 3 (inequalities (A.2) and (A.3)), we can obtain the following chain

$$\|q'\|_1 \ge \|q'\|_0 \ge C\|q\|_*,$$

which is equivalent to the desired inequality (A.1).

The following statement will be also of some use for our purpose.

Lemma 4. *For any function* $q \in \mathbf{H}_0^1[0,1]$ *the following estimate holds*

$$|q(1)| \le \sqrt{2}\|q\|_1. \tag{A.6}$$

Proof. Since $q \in \mathbf{H}_0^1[0,1]$, $q(0) = 0$ and by integrating by parts, we obtain

$$q^{2(1)} = 2\int_0^1 q(\sigma)q'(\sigma)\mathrm{d}\sigma$$

Straightforwardly applying the Cauchy-Bunyakovskii inequality and Lemma 2, we arrive at the desired inequality (A.6).

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